6. The amount of absorption produced by a given thickness of a solution of a metallic salt is not proportional to the amount of salt in solution, but appears to follow approximately a logarithmic law.
7. The amount of absorption varies logarithmically with the thickness of the solution traversed by the rays. The percentage absorption may be represented by an equation of the form $r = \log(\lambda t + \mu)$ where λ is a constant depending on the nature of the solution, and on the penetrative power of the X rays, and t is the thickness of the solution traversed.

It is to be noticed that all the above tables and conclusions are based on measurements made with solutions containing the equivalent weight of the salts, in grams per litre, so that the molecular absorption of the salts with divalent or trivalent bases is greater than that indicated in the tables.

The conclusions 1, 5, and 7, given above, are in exact agreement with those stated by Dr. Gladstone and Mr. Hibbert in their articles in the 'Chemical News' (*loc. cit.*).

Numerous references have been made throughout this paper to their articles and reports before the British Association, though it was not until the above investigations were nearly completed that attention was drawn to the work that had been done by them.

"On the Resistance to Torsion of certain Forms of Shafting, with special Reference to the Effect of Keyways."

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(Abstract.)

The object of the present paper is to obtain solutions of the problem of Torsion for certain cylinders, whose cross-sections are bounded by confocal conics. It is mainly an extension of de Saint-Venant's investigations, and is based upon his general equations of torsion.

The method employed depends upon the use of conjugate functions ξ and η, such that ξ = const. represents confocal ellipses and η = const. confocal hyperbolas.

The use of conjugate functions for the torsion problem has been suggested by Thomson and Tait,* by Clebsch,† and by Boussinesq.‡

* 'Natural Philosophy.'

† 'Theorie der Elasticität fester Körper,' §§ 33—35.

‡ 'Journal de Mathématiques,' pp. 177—186, série 3, vol. 6.

Clebsch has used such elliptic co-ordinates to solve the torsion problem for hollow cylinders bounded by confocal ellipses, and de Saint-Venant has applied conjugate functions to the same problem for shafts whose sections are sectors of circles; curvilinear co-ordinates have also been employed by Mr. H. M. Macdonald,* but I am not aware that the actual solution has yet been obtained for sections bounded by both ellipses and hyperbolas.

The work proceeds on lines analogous to those developed by Saint-Venant himself in his solution of the problem of torsion for the cylinder of rectangular cross-section. The strains and stresses are expressible in terms of infinite series involving circular and hyperbolic functions.

The boundaries of the section are given by constant values of ξ and η. The values of ξ are taken to be $\pm\, \alpha$.

The conditions from which the unknown quantity w (the shift parallel to the axis) is determined are

$$d^2w/dx^2 + d^2w/dy^2 = 0$$

throughout the sections; and

$$dw/dn + (mx - ly)\tau = 0$$

along the boundary, where dn = an element of the outwards normal to the boundary, τ is the angle of torsion per unit length, and l, m are the direction-cosines of dn.

Now in the present case

$$dn = \pm\, d\xi \times (c\sqrt{\mathrm{J}})$$

where $\mathrm{J} = \partial\left(\frac{x}{c}, \frac{y}{c}\right)\Big/\partial(\xi, \eta,)$ at the boundary where ξ = const. and

$$dn = \pm\, d\eta \times (c\sqrt{\mathrm{J}})$$

at the boundary where η = const., the sign being determined so that dn is positive.

By adding suitable terms to w, we can reduce one or other of the boundary conditions to the form

$$dw_1/dn = 0$$

where $w = w_1$ + suitable terms.

Suppose we make

$$dw_1/d\xi = 0\,;\ \xi = \pm\,\alpha.$$

Expanding now w_1 in the form of a series,

$$w_1 = \sum_{n=0}^{n=\infty} \mathrm{A}_n \sin h \left\{ \frac{m+1}{2\alpha}\pi(\eta + \kappa) \right\} \sin \frac{\overline{m+1}\,\pi\xi}{2\alpha},$$

* "On the Torsional Strength of a Hollow Shaft," 'Proc. Camb. Phil. Soc.,' vol. 8, 1893, pp. 62 *et seq.*

the differential equation and the first boundary condition are identically satisfied.

When this value is substituted in the second boundary condition, we get an equation expressing a given function of ξ in a series of sines of odd multiples of $\pi\xi/2\alpha$, between the limits $+\alpha$ and $-\alpha$.

But such an expression can be definitely obtained by a method analogous to that for Fourier's series. Comparing coefficients, we obtain relations which determine completely all the constants in the expression of w_1.

w is then known. The shears and torsion moment are then deduced by differentiation and a double integration.

The cross-sections which are dealt with in the present paper, are of very great generality, and they include as special cases many of the cross-sections which Saint-Venant has worked out, for instance the rectangle and the sector of a circle.

The first section of which I treat is that bounded by an ellipse and two confocal hyperbolas. Although the analysis is worked out for the case where the two hyperbolic segments are not symmetrical, I have not given any numerical examples of this case, as the sections obtained by taking two hyperbolas curved the same way do not correspond to any interesting practical case: the section is too broad at the ends and too narrow at the bend, to be any fair representation of the angle iron.

The section bounded by an ellipse and the two branches of a confocal hyperbola is, on the other hand, an approximate representation of a well-known section, much used in engineering practice, the rail section. This section I have worked out for various values of the eccentricity of the ellipse, and of the angle between the asymptotes of the hyperbola. The four sections where this angle is 120° give the best representation of the rail section.

The numerical results are tabulated so as to show the ratio of the torsional rigidity of this section to that of the circular section of the same area, and also the same ratio for the maximum stress.

The ratio of these two ratios gives us a kind of measure of the usefulness or "efficiency" of the section.

In the case of the latter class of sections, I have investigated at length the position of the *fail-points*, or points of maximum strain and stress, the maximum strain, in the case of torsion, being coincident with the maximum stress. It is found that for the two smaller ellipses the maximum stress occurs at the point B, where the section is thinnest. For the two larger ellipses, the maximum stress occurs at four points, F, F, F, F, symmetrically distributed round the contour, and lying on the broad sides of the section. The critical section, when these two cases pass into one another, can be calculated, and is shown in the paper.

The changes in the stresses are shown by curves accompanying the paper, in which the abscissa represents the quantity α, whose hyperbolic cosine and sine are proportional to the major and minor axes of the ellipse respectively, and in which the ordinates represent the stresses at A, B, F, divided by the maximum stress of the circular section of equal area. The curves are, in certain parts, only roughly drawn, but they suffice to show the manner in which the stresses vary. It is seen that the stress at B separates from the maximum stress after the critical value $\alpha = 1\cdot225$, and gradually diminishes, compared with stresses at A and F.

This result might have been expected from the investigations of de Saint-Venant upon certain sections bounded by curves of the fourth degree.

These investigations appear, however, not to have been sufficiently noticed. Thomson and Tait in their 'Natural Philosophy,' and Boussinesq in his researches on Torsion,* both conclude that the fail-points are at the points of the cross-section nearest to the centre, and Boussinesq even gives an apparently general proof of this proposition. His proof, however, is subject to certain restrictions, which I point out, and which prevent it from being applied to the sections I am dealing with.

The sections are sensibly less useful than the circular section, their torsional rigidity being always diminished and the maximum stress very often increased. This remark, I may add, applies to all the sections dealt with in this paper.

This usefulness or efficiency decreases as the neck of the section becomes more narrow, as indeed might have been anticipated.

Other sections worked out are those corresponding to angles between the asymptotes of 90°, 60°, and 0°; in the latter case the sections degenerate into ordinary elliptic sections, with two straight slits or indefinitely thin keyways, cut into them along the major axis, as far as the foci. The stress at the foci, however, is then theoretically infinite.

It is interesting to see how, as we make the bend round the foci sharper, the values of α, for which the two fail-points break up into four, become larger and larger, until, when the angle between the asymptotes of the hyperbolas is less than 73°, the greatest stress always occurs at the neck of the section.

The limiting case of such sections, when the angle between the asymptotes is very small, and the eccentricity of the ellipse nearly unity, the distance between the foci being very great, gives us the rectangle.

I then pass on to the section bounded by one ellipse and one confocal hyperbola. In the limiting case, when the foci coincide, we obtain the sector of a circle. Of this I have worked out numerically three cases, in each case taking two ellipses.

1. The semi-ellipse.

* 'Journal de Mathématiques,' série 2, vol 16, p. 200.

2. The ellipse with a keyway cut into it of the shape of a rectangular confocal hyperbola.

3. The ellipse with a single slit cut into it.

The most striking of the results is in reference to the reduction of the torsional rigidity of the ellipse in case (3). This reduction of rigidity decreases rapidly as the depth of the notch decreases.

The rigidity, which is reduced by as much as 23 per cent. when the depth of the keyway is as great as 0·6 (semi-major axis), falls to about 1 per cent. when this depth is 0·12 (semi-major axis).

Possibly this may throw some light on the fact that the effect of cutting such slits into the material does not always give in practice the reduction in the torsional rigidity which should have been expected from Saint-Venant's results for the circle. Clearly the depth of the keyway is a factor of the very first importance, and keyways of moderate depth will produce a comparatively small effect on the torsional rigidity.

It is also shown that the effect of cutting two equal and opposite slits is practically equal, in the two cases which I have calculated (namely $\alpha = \pi/6$ and $\alpha = \pi/2$), to twice the effect of a single slit.

It seems, therefore, that the study of these sections brings to light several interesting facts in the theory of elasticity, and will well repay the trouble involved in dealing with the long and somewhat tedious algebra and arithmetic which lead to these results.

November 16, 1899.

The LORD LISTER, F.R.C.S., D.C.L., President, in the Chair.

Professor W. F. Barrett and Mr. J. S. Gamble were admitted into the Society.

A List of the Presents received was laid on the table, and thanks ordered for them.

In pursuance of the Statutes, notice of the ensuing Anniversary Meeting was given from the Chair.

Professor Meldola, Professor Perry, and Dr. R. H. Scott were by ballot elected Auditors of the Treasurer's accounts on the part of the Society.

The following Papers published during the recess, in full or in abstract, in accordance with the Standing Orders of Council, were read in title :—